Assignment 10

Deadline: April 6, 2018.

Hand in: Supp. Ex. no 2, 3.

Supplementary Exercise

1. (a) Show that
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}dt \ .$$
Suggestion: Think about
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \frac{(-x)^n}{1+x} \ .$$
(b) Show that
$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \le \frac{x^{n+1}}{n+1} \ .$$

2. This exercise suggests an alternative way to define the logarithmic and exponential functions. Define $\text{nog} : (0,\infty) \to \mathbb{R}$ by
$$\text{nog}(x) = \int_1^x \frac{1}{t}dt.$$
(a) $\text{nog}(x)$ is strictly increasing, concave, and tends to ∞ and $-\infty$ as $x \to \infty$ and 0 respectively.
(b) $\text{nog}(xy) = \text{nog}(x) + \text{nog}(y)$.
(c) Define $e(x)$ to be the inverse function of nog. Show that it coincides with $E(x)$.

Note: f is concave means $-f$ is convex. You cannot assume $\log x$ has been defined.

3. (a) Show that there is a unique solution $c(x), x \in \mathbb{R}$, to the problem
$$f'' = f, \quad f(0) = 1, \ f'(0) = 0.$$
(b) Letting $s(x) \equiv c'(x)$, show that s satisfies the same equation as c but now $s(0) = 0, \ s'(0) = 1$.
(c) Establish the identities, for all x,
$$c^2(x) - s^2(x) = 1,$$
and
$$c(x+y) = c(x)c(y) + s(x)s(y).$$
(d) Express c and s as linear combinations of e^x and e^{-x}. (c and s are called the hyperbolic cosine and sine functions respectively. The standard notations are $\cosh x$ and $\sinh x$. Similarly one can define other hyperbolic trigonometric functions such as $\tanh x$ and $\coth x$.)